8 November 2013, 8:30 – 11:30

Rijksuniversiteit Groningen
Statistiek

Tentamen

RULES FOR THE EXAM:

- The use of a normal, non-graphical calculator is permitted.
- This is a CLOSED-BOOK exam.
- At the end of the exam you can find a normal table and a chi-squared table.
- Your exam mark : 10 + 90×your score/75 .

1. **Rao-Blackwell: Improving estimators.** Let X be observed data. Let $\hat{\theta}(X)$ be an unbiased estimate of θ and let T be a sufficient statistic for θ. Define the new estimator $\hat{\theta}^*$ of θ,

$$\hat{\theta}^*(X) = E(\hat{\theta}(X)|T).$$

   Then show that

   (a) Show that $\hat{\theta}^*(X)$ is a valid estimator, i.e. a function of the data that does not depend on θ. [5 Marks]

   (b) Show that $\hat{\theta}^*(X)$ is unbiased. [5 Marks]

   (c) Show that $\hat{\theta}^*(X)$ has a variance that is lower than (or equal to) that of $\hat{\theta}$. [5 Marks]

2. **Linear regression.** Let $Y \in \mathbb{R}^n$ be independent observations on n subjects. Let X be a $n \times (p+1)$ matrix, such that

$$Y \sim N(X\beta, \sigma^2 I_n),$$

   where $(\beta = (\beta_0, \beta_1, \ldots, \beta_p), \sigma^2) \in \mathbb{R}^{p+2}$ are unknown coefficients and I_n is the $n \times n$ identity matrix.

   (a) Derive the maximum likelihood estimate $\hat{\beta}$ of β. [5 Marks]

   (b) Determine whether $\hat{\beta}$ is unbiased. [5 Marks]

   (c) Derive the variance of $\hat{\beta}$. [5 Marks]

   (d) In a particular case, we have two predictors, i.e. $p = 3$, and we want to test whether $\beta_1 = \beta_2 = 0$ using the likelihood ratio test. The transformed likelihood ratio statistic Λ is calculated to be

$$-2\log(\Lambda) = 4.1.$$

   Test at the 5% level whether or not to reject the null-hypothesis. [5 Marks]

3. **Point estimation.** Let $X_1, \ldots, X_n$ be a sample of independent, identically distributed random variables, with density

$$f_\theta(x) = \begin{cases} \theta x + \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

for $\theta \in (-\frac{1}{2}, \frac{1}{2})$. Let

$$\hat{\theta}_n = 3\bar{X}/2$$

be an estimator of θ, where $\bar{X}$ is the sample mean.

(a) Determine whether $\hat{\theta}_n$ is unbiased. [5 Marks]

(b) Determine whether $\hat{\theta}_n$ is consistent. [5 Marks]

(c) Show that $\hat{\theta}_n$ is not sufficient [Hint: give a specific example]. [5 Marks]

(d) Determine whether $\hat{\theta}_n$ is efficient. [5 Marks]

4. **Optimal testing.** Consider a single observation X from a geometric distribution, $X \sim \text{Geometric}(p)$, i.e. with density

$$f_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots,$$

and cumulative distribution function

$$F_X(k) = 1 - (1-p)^k, \quad k = 1, 2, \ldots.$$

We want to test the following hypotheses:

$$H_0: \quad p = 0.10$$
$$H_1: \quad p = 0.05$$

(a) We want to perform an optimal test with a significance level of at most 5% of the null hypothesis against the alternative. Determine the critical region. [15 Marks]

(b) What is the power of this test? [5 Marks]

Below a statistical table which may be used in the calculations.

$\nu \setminus \alpha$	0.995	0.99	0.975	0.95	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	11.070	12.833	15.086	16.750
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188

Table 1: Values of $\chi^2_{\alpha,\nu}$ as found in the book: the entries in the table correspond to values of x, such that $P(\chi^2_\nu > x) = \alpha$, where χ^2_ν correspond to a chi-squared distributed variable with ν degrees of freedom.